Bayesian Mechanism Design

Jason D. Hartline

Northwestern University

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Vignettes from Manuscript

Mechanism Design and Approximation

http://jasonhartline.com/MDnA/

Mechanism Design

Basic Mechanism Design Question: How should an economic system be designed so that selfish agent behavior leads to good outcomes?

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General Theme: resource allocation.

Overview

Part I: Optimal Mechanism Design

- single-item auction.
- objectives: social welfare vs. seller profit.
- characterization of Bayes-Nash equilibrium.
- consequences: solving and optimizing over BNE.
- inferring values from bids.

Part II: Approximation in Mechanism Design

- single-item auctions.
- multi-dimensional auctions.
- prior-independent auctions.
- computationally tractable mechanisms.

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Part II: Approximation in Mechanism Design

- single-item auctions. (Chapter 4)
- multi-dimensional auctions. (Chapter 7)
- prior-independent auctions. (Chapter 5)
- computationally tractable mechanisms. (Chapter 8)

Single-item Auction

Mechanism Design Problem: *Single-item Auction*

Given:

- one item for sale.
- n bidders (with unknown private values for item, $v_1, \ldots, v_n$)
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- Auction to solicit bids and choose winner and payments.

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Possible Auction Objectives:

- Maximize *social surplus*, i.e., the value of the winner.
- Maximize *seller profit*, i.e., the payment of the winner.

Objective 1: maximize social surplus

Example Auctions

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Questions:

- what are equilibrium strategies?
- what is equilibrium outcome?
- which has higher surplus in equilibrium?
- which has higher profit in equilibrium?

Second-price Auction Equilibrium Analysis

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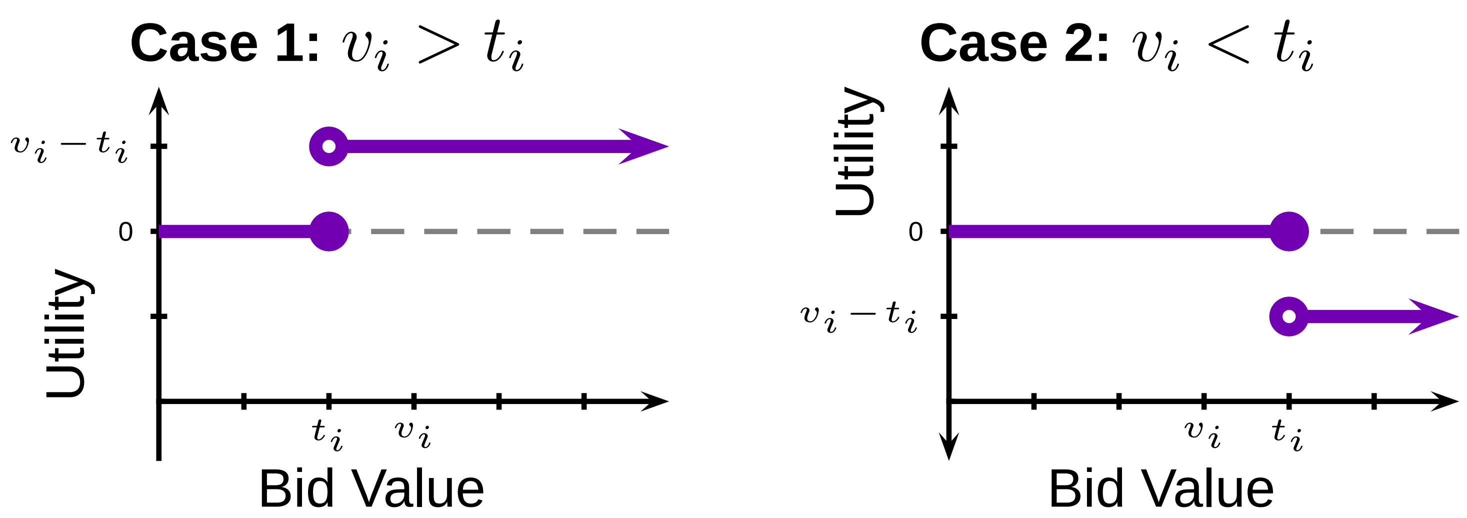

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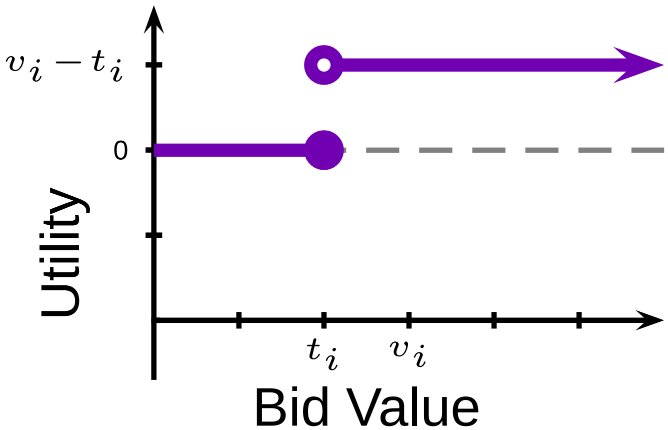

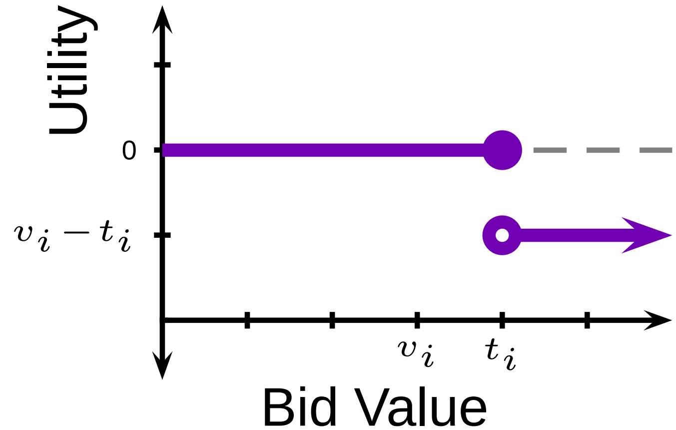

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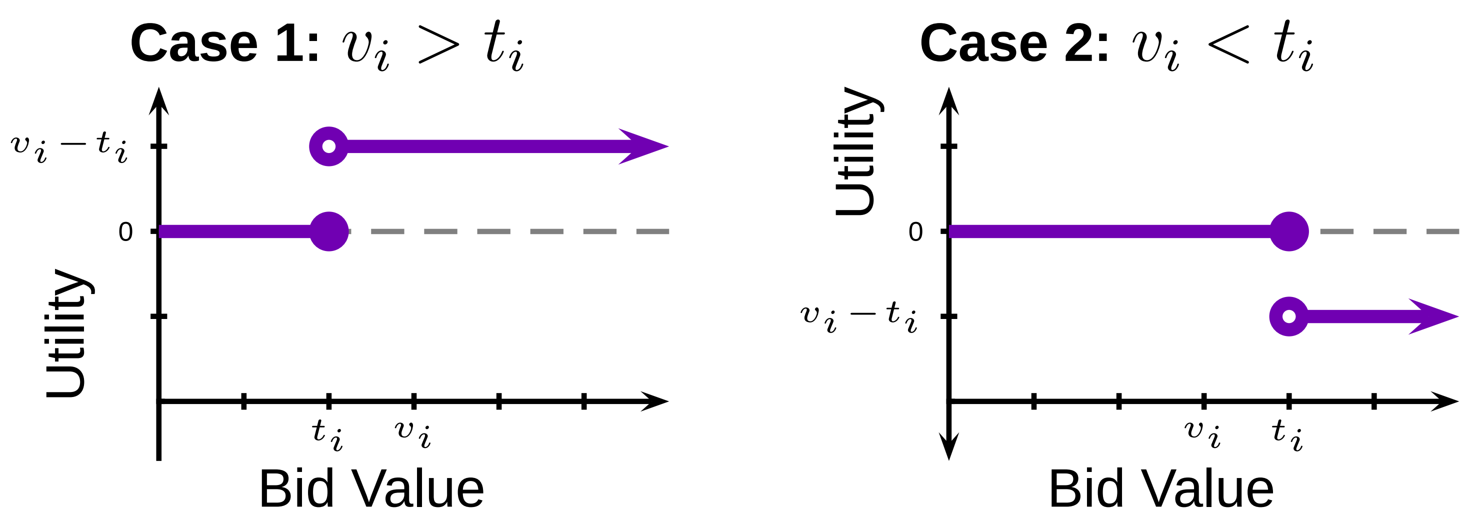

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What about first-price auction?

Recall First-price Auction

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Note: first-price auction has no DSE.

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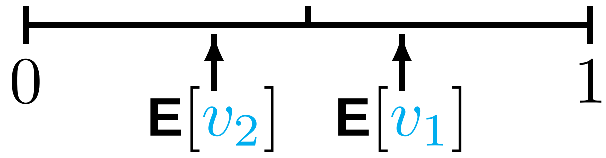

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Conclusion 2: bidder with highest value wins

Conclusion 3: first-price auction maximizes social surplus!

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Definition: a *strategy profile* is in *Bayes-Nash Equilibrium (BNE)* if for all i, $s_i(v_i)$ is best response when others play $s_j(v_j)$ and $v_j \sim F_j$.

Conclusions:

- second-price auction maximizes surplus in DSE regardless of distribution.
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Surplus Maximization Conclusions

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Questions?

Objective 2: maximize seller profit

(other objectives are similar)

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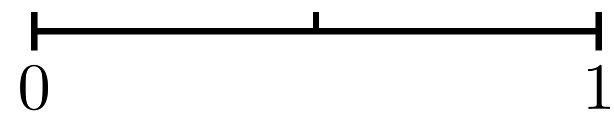

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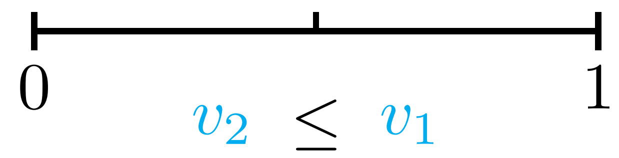

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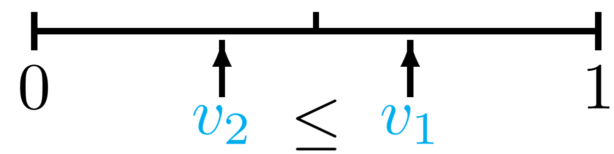

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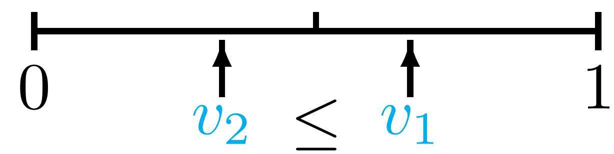

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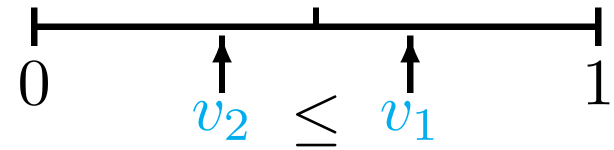

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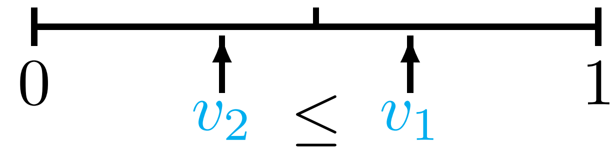

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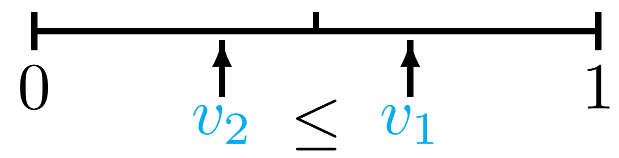

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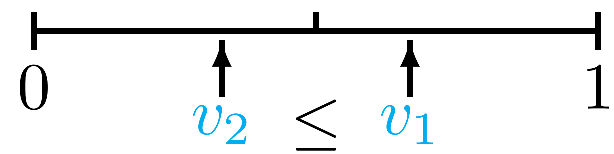

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Surprising Result: second-price and first-price auctions have same expected profit.

Can we get more profit?

Second-price with reserve price

Second-price Auction with reserve r

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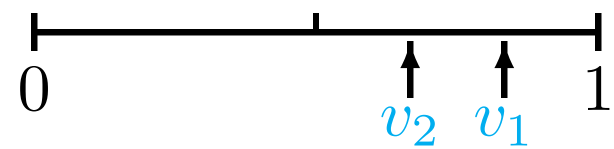

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Case 2: $v_1 \geq v_2 \geq \frac{1}{2}$	$1/4$	**E**$[v_2 \mid \text{Case } 2] = \frac{2}{3}$
Case 3: $v_1 \geq \frac{1}{2} > v_2$	$1/2$	$\frac{1}{2}$

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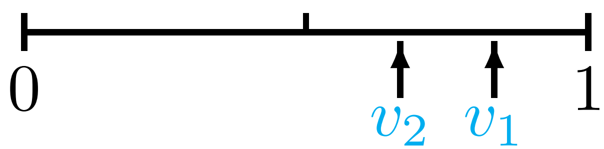

- Sort values, $v_1 \geq v_2$

Case Analysis:	**Pr**[Case i]	**E**[Profit]
Case 1: $\frac{1}{2} > v_1 \geq v_2$	$1/4$	0
Case 2: $v_1 \geq v_2 \geq \frac{1}{2}$	$1/4$	**E**$[v_2 \mid \text{Case } 2] = \frac{2}{3}$
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E[profit of 2nd-price with reserve] $= \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{1}{2} = \frac{5}{12}$

Second-price Auction with reserve r

0. Insert seller-bid at r. 1. Solicit bids. 2. Winner is highest bidder. 3. Charge 2nd-highest bid.

Lemma: Second-price with reserve r has truthful DSE.

What is profit of Second-price with reserve $\frac{1}{2}$ on two bidders $U[0,1]$?

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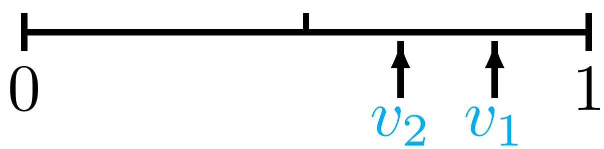

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Profit Maximization Observations

Observations:

- pretending to value the good increases seller profit.
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Bayes-Nash Equilibrium Characterization and Consequences

0. characterization.

1. solving for BNE.

2. optimizing over BNE.

Notation:

- $\mathbf{x}$ is an allocation, x_i the allocation for i.
- $\mathbf{x}(\mathbf{v})$ is BNE allocation of mech. on valuations $\mathbf{v}$.
- $\mathbf{v}_{\text{-}i} = (v_1, \ldots, v_{i-1}, ?, v_{i+1}, \ldots, v_n)$.

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Analogously, define $\mathbf{p}$, $\mathbf{p}(\mathbf{v})$, and $p_i(v_i)$ for payments.

Characterization of BNE

Thm: a mechanism and strategy profile is in BNE iff

1. *monotonicity (M)*: $x_i(v_i)$ is monotone in v_i.
2. *payment identity (PI)*: $p_i(v_i) = v_i x_i(v_i) - \int_0^{v_i} x_i(z)dz + p_i(0)$.

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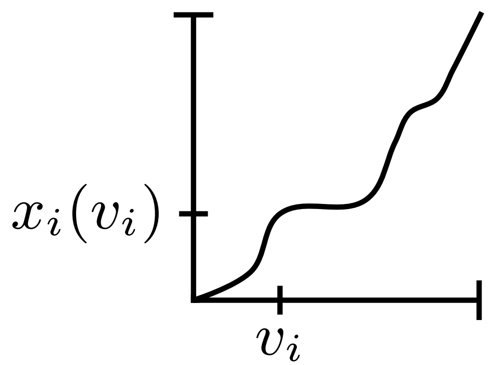

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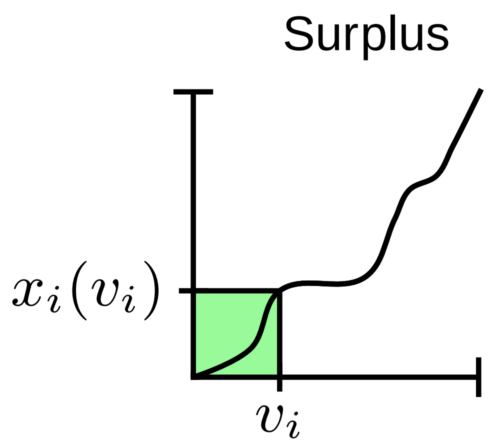

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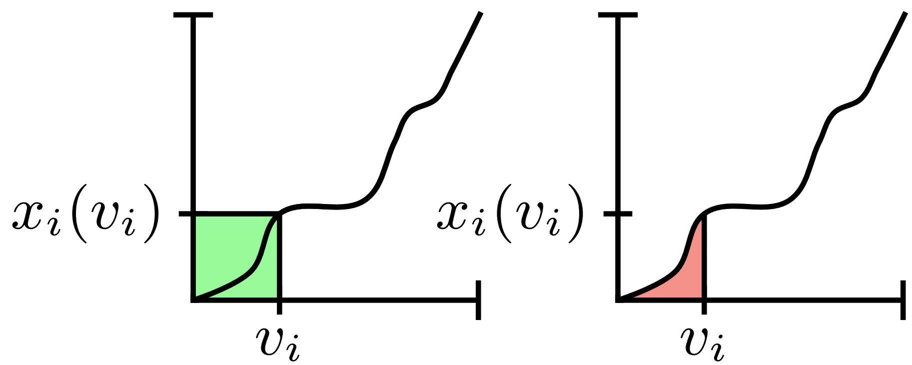

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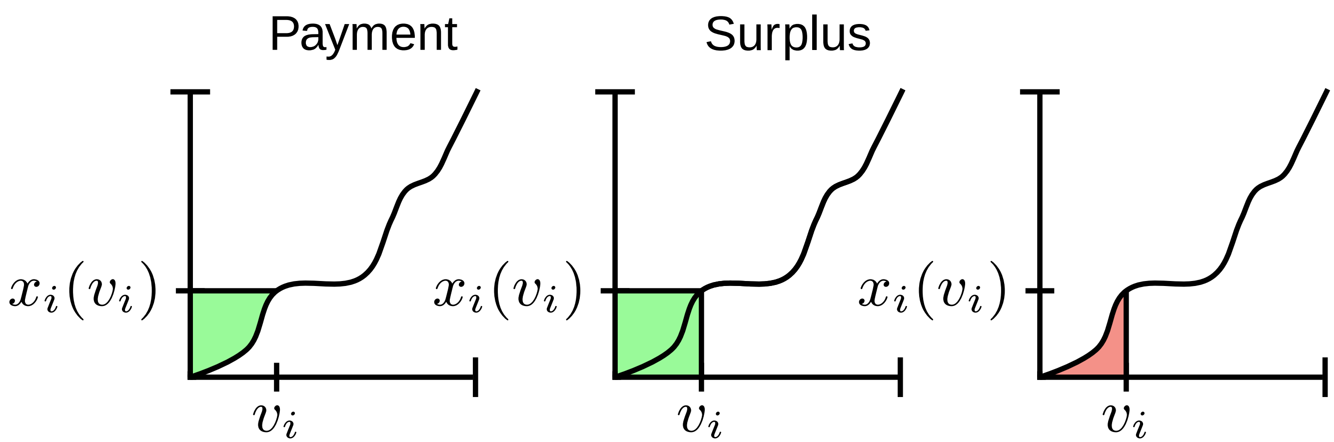

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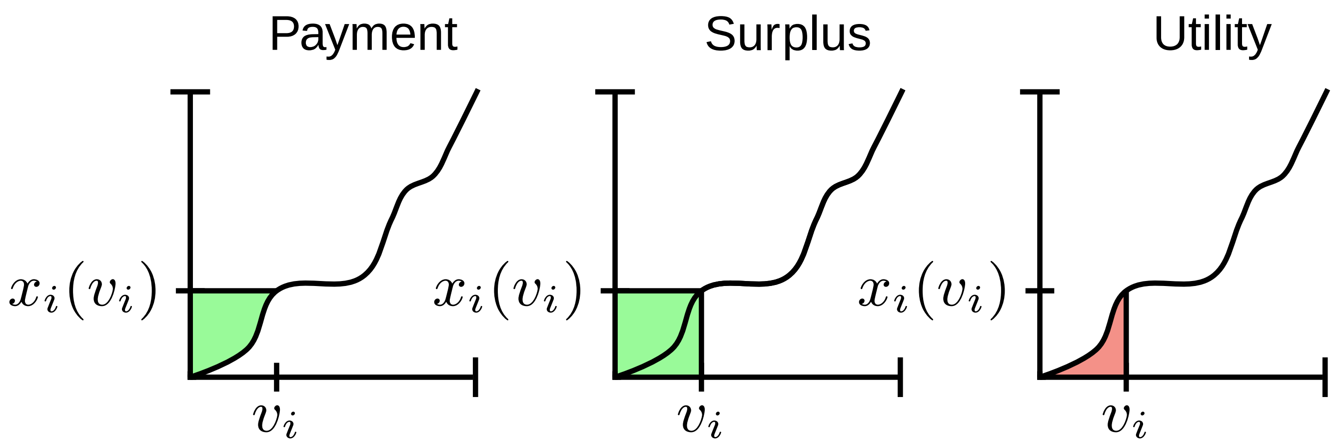

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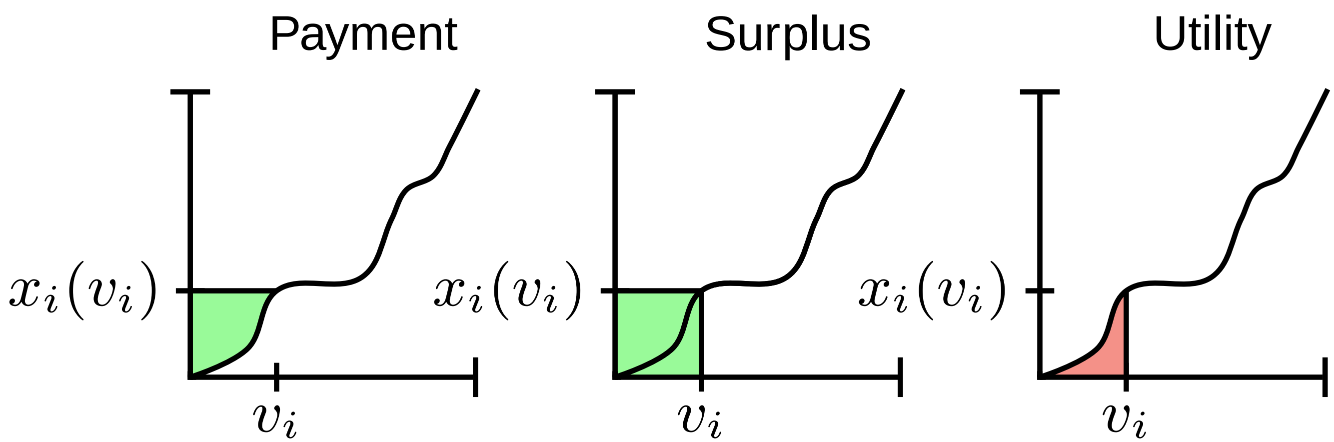

Consequence: *(revenue equivalence)* in BNE, auctions with same outcome have same revenue (e.g., first and second-price auctions)

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Solving for equilbrium:

1. What happens in first-price auction equilibrium?

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3. Verify guess and BNE: $b(v)$ continuous, strictly increasing, symmetric.

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Proof: expected virtual valuation of winner $=$ expected payment.

Proof of Lemma

Recall Lemma: In BNE, $\mathbf{E}[p_i(v_i)] = \mathbf{E}\left[\left(v_i - \frac{1-F_i(v_i)}{f_i(v_i)}\right) x_i(v_i)\right]$.

Proof Sketch:

- Use characterization: $p_i(v_i) = v_i x_i(v_i) - \int_0^{v_i} x_i(v)dv$.
- Use definition of expectation (integrate payment $\times$ density).
- Swap order of integration.
- Simplify.

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What is optimal single-item auction for $U[0, 1]$?

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- So, optimal auction is Second-price Auction with reserve 1/2!

Conclusions:

- expected virtual value $=$ expected revenue
- optimal mechanism maximizes virtual surplus.
- optimal auction depends on distribution.
- i.i.d., regular distributions: second-price with reserve is optimal.
- theory is “descriptive”.

Questions?

Inferring Values from Bids.

Auction Design Challenge

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Data: bids and revenues (for 200 auctions)

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* all data is synthetic; counter-factuals known.

The Data

Auction	Bid 1	Bid 2	Revenue
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2	0.11	0.42	0.42
3	0.08	0.86	0.86
4	0.50	0.48	0.50
5	0.69	0.83	0.83
6	0.46	0.58	0.58
7	0.53	0.03	0.53
8	0.77	0.60	0.77
9	0.91	0.49	0.91
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2. with reserve price, bidders should raise their bids.

The Data

Auction	Bid 1	Bid 2	Revenue
1	0.74	0.34	0.74
2	0.11	0.42	0.42
3	0.08	0.86	0.86
4	0.50	0.48	0.50
5	0.69	0.83	0.83
6	0.46	0.58	0.58
7	0.53	0.03	0.53
8	0.77	0.60	0.77
9	0.91	0.49	0.91
10	0.54	0.50	0.54
11	0.44	0.35	0.44
⋮	⋮	⋮	⋮
200	0.44	0.54	0.54
Average			0.68

Failed Approach: simulate reserve prices with old bid data.

Discussion:

1. loss in revenue when bids below reserve.
2. with reserve price, bidders should raise their bids.

Problem: simulation does not account for bidders raising bids!

Behavior vs. Simulations

Auction	Bid 1	Bid 2	Revenue
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Behavior vs. Simulations

Auction	Bid 1	Bid 2	Revenue	Sim 0.5
1	0.74	0.34	0.74	0.74
2	0.11	0.42	0.42	0.00
3	0.08	0.86	0.86	0.86
4	0.50	0.48	0.50	0.00
5	0.69	0.83	0.83	0.83
6	0.46	0.58	0.58	0.58
7	0.53	0.03	0.53	0.53
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10	0.54	0.50	0.54	0.54
11	0.44	0.35	0.44	0.00
⋮	⋮	⋮	⋮	⋮
200	0.44	0.54	0.54	0.54
Average			0.68	0.60

Behavior vs. Simulations

Auction	Bid 1	Bid 2	Revenue	Sim 0.5	Real 0.5
1	0.74	0.34	0.74	0.74	0.83
2	0.11	0.42	0.42	0.00	0.57
3	0.08	0.86	0.86	0.86	0.93
4	0.50	0.48	0.50	0.00	0.62
5	0.69	0.83	0.83	0.83	0.91
6	0.46	0.58	0.58	0.58	0.69
7	0.53	0.03	0.53	0.53	0.65
8	0.77	0.60	0.77	0.77	0.85
9	0.91	0.49	0.91	0.91	0.98
10	0.54	0.50	0.54	0.54	0.65
11	0.44	0.35	0.44	0.00	0.58
⋮	⋮	⋮	⋮	⋮	⋮
200	0.44	0.54	0.54	0.54	0.66
Average			0.68	0.60	0.76

Behavior vs. Simulations

Auction	Bid 1	Bid 2	Revenue	Sim 0.5	Real 0.5	Sim 0.75	Real 0.75
1	0.74	0.34	0.74	0.74	0.83	0.00	0.93
2	0.11	0.42	0.42	0.00	0.57	0.00	0.76
3	0.08	0.86	0.86	0.86	0.93	0.86	1.02
4	0.50	0.48	0.50	0.00	0.62	0.00	0.78
5	0.69	0.83	0.83	0.83	0.91	0.83	1.00
6	0.46	0.58	0.58	0.58	0.69	0.00	0.82
7	0.53	0.03	0.53	0.53	0.65	0.00	0.80
8	0.77	0.60	0.77	0.77	0.85	0.77	0.95
9	0.91	0.49	0.91	0.91	0.98	0.91	1.06
10	0.54	0.50	0.54	0.54	0.65	0.00	0.80
11	0.44	0.35	0.44	0.00	0.58	0.00	0.76
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
200	0.44	0.54	0.54	0.54	0.66	0.00	0.80
Average			0.68	0.60	0.76	0.38	0.85

Behavior vs. Simulations (cont.)

Reserve	Behavior	Simulation
0.0	0.68	0.68

Behavior vs. Simulations (cont.)

Reserve	Behavior	Simulation
0.0	0.68	0.68
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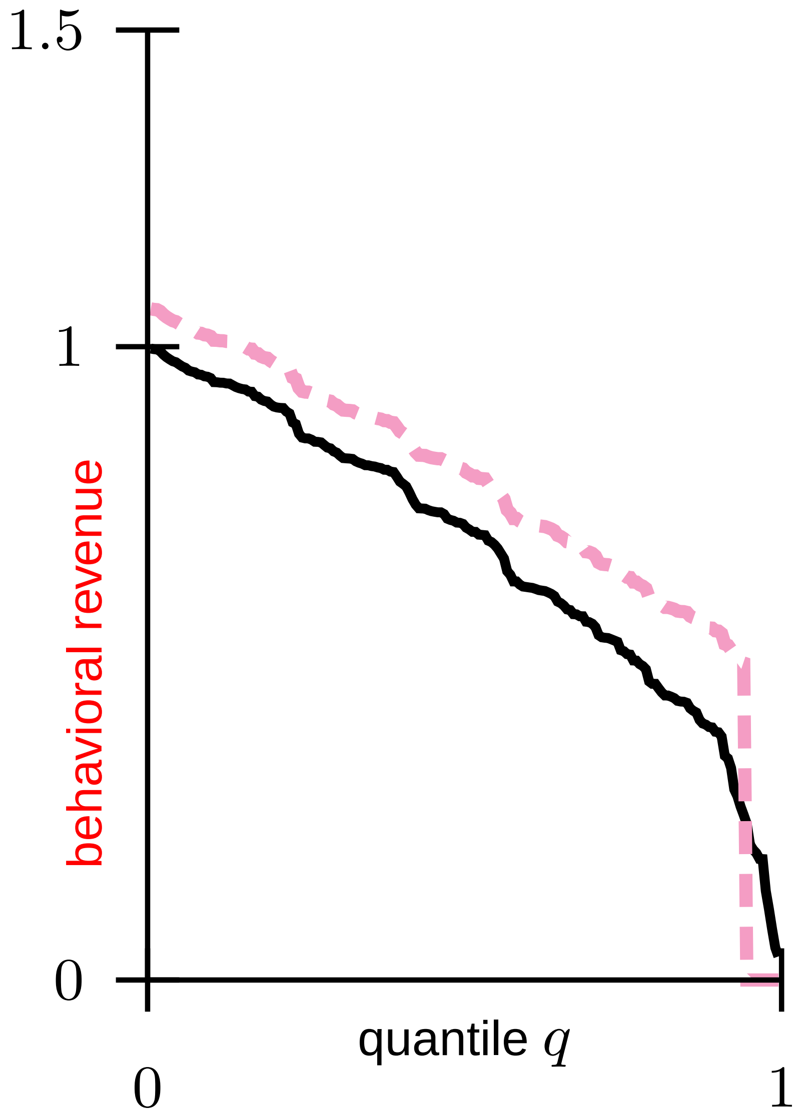

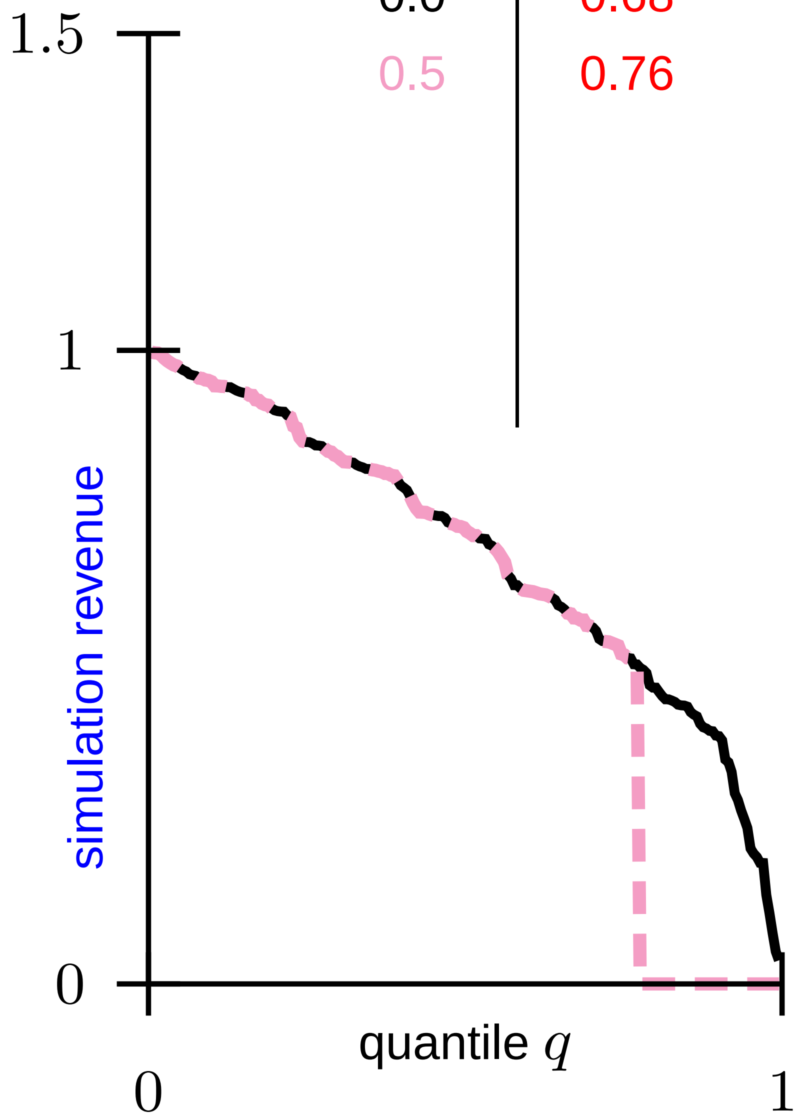

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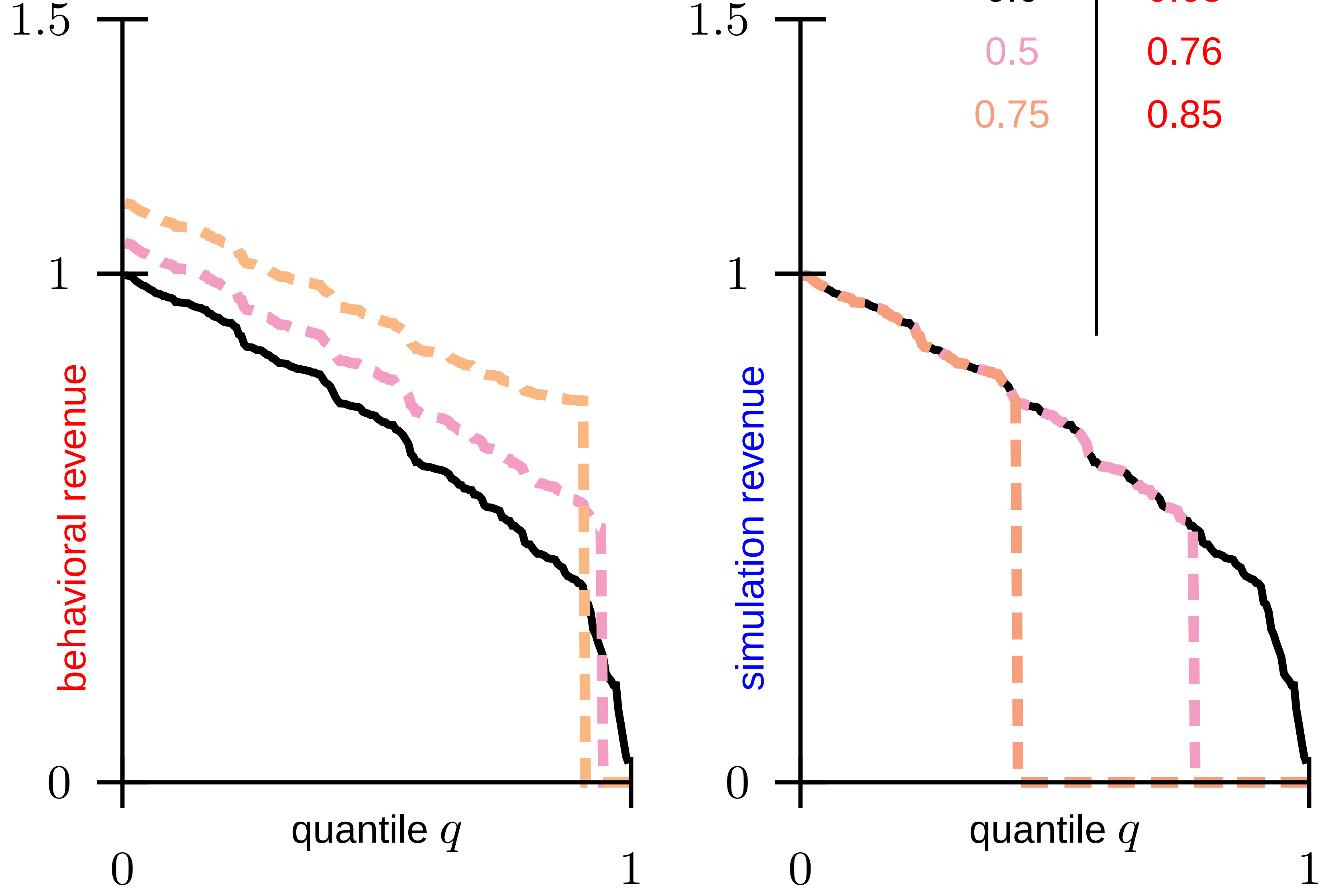

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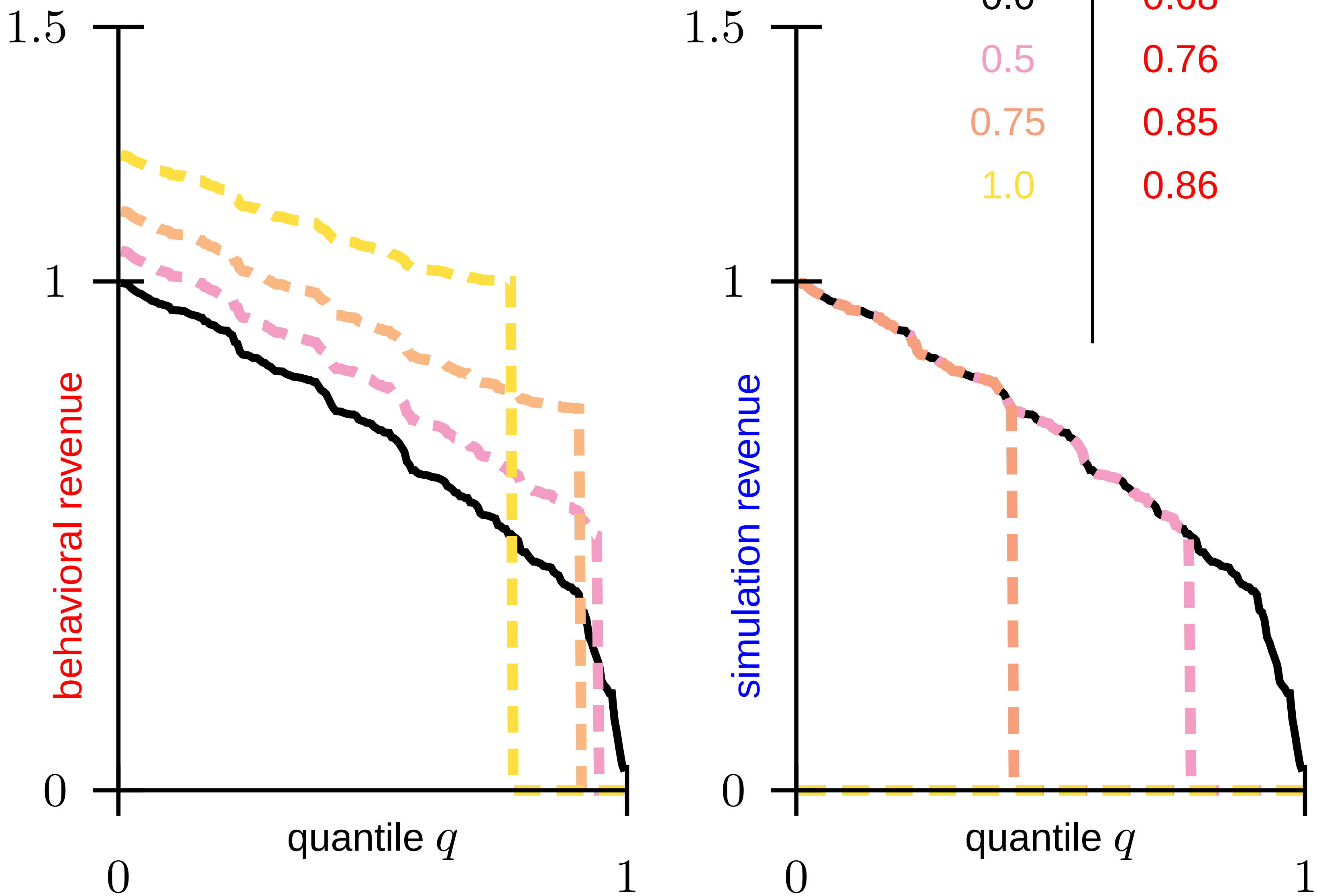

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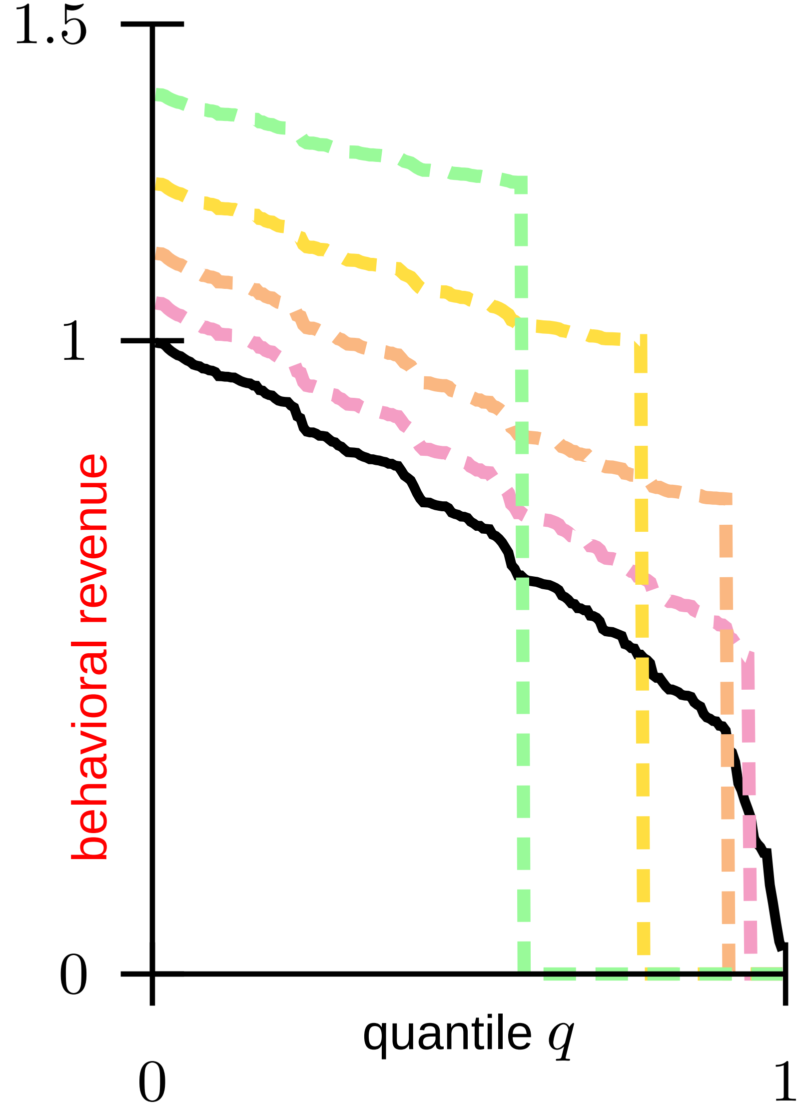

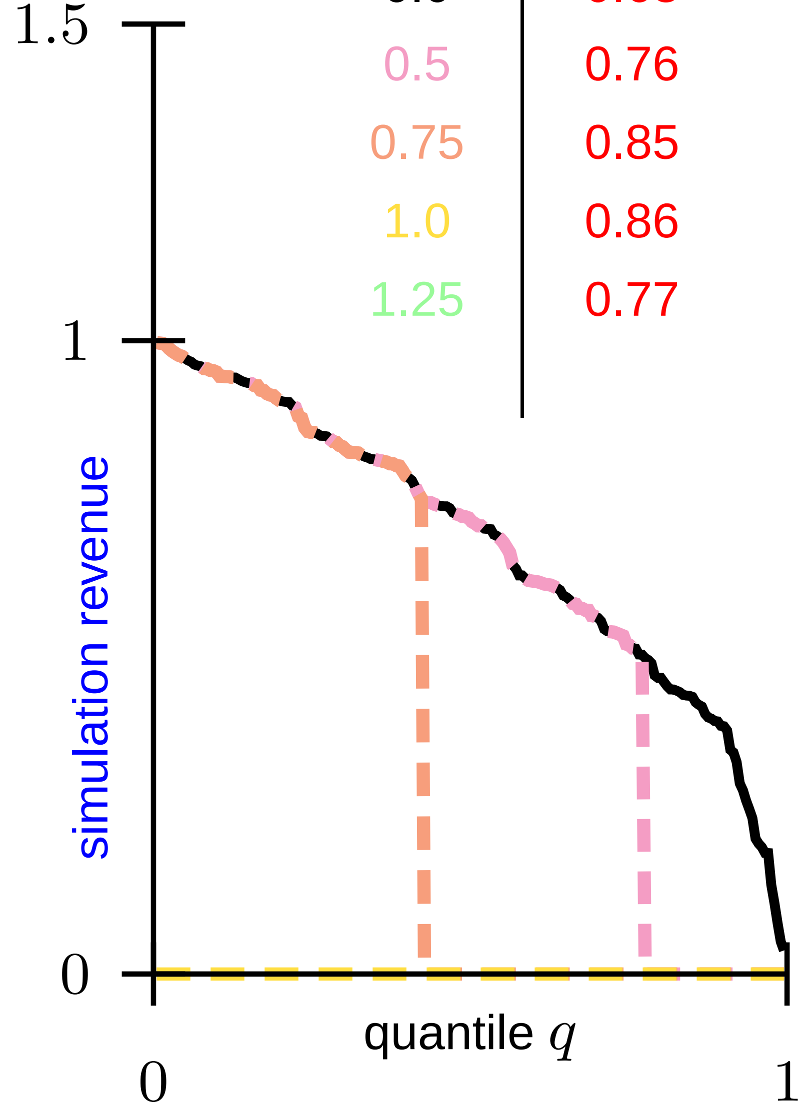

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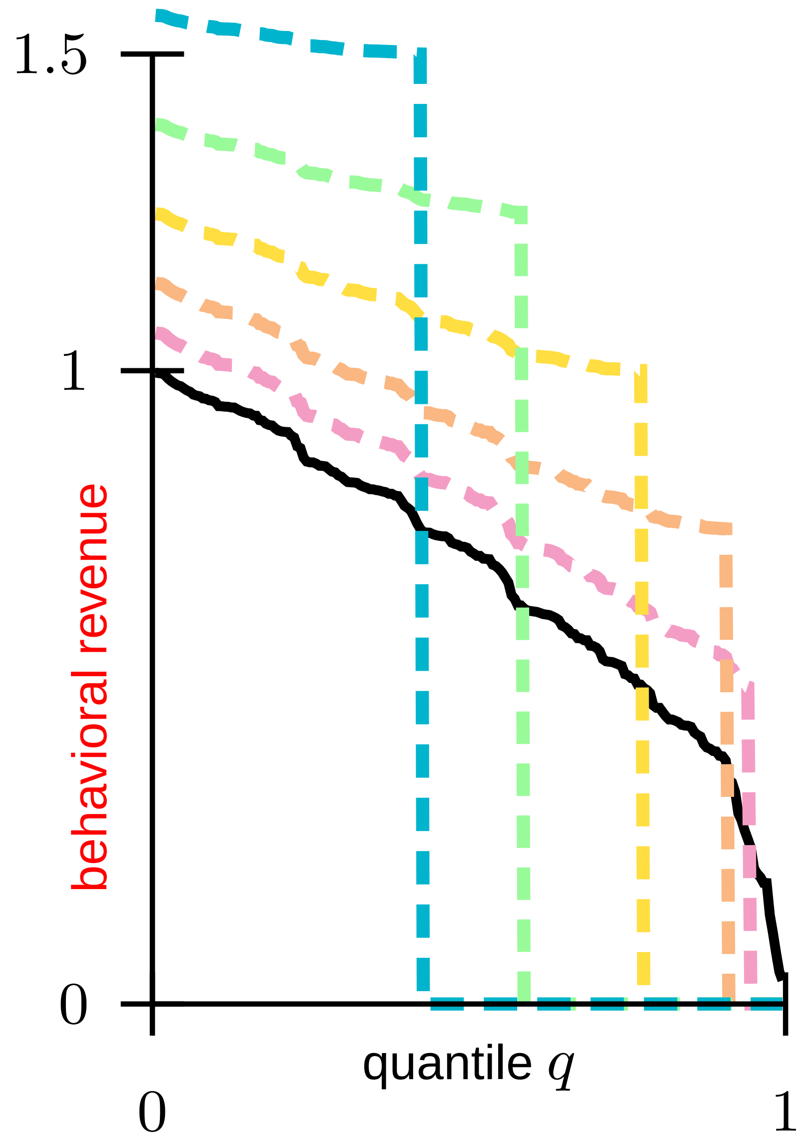

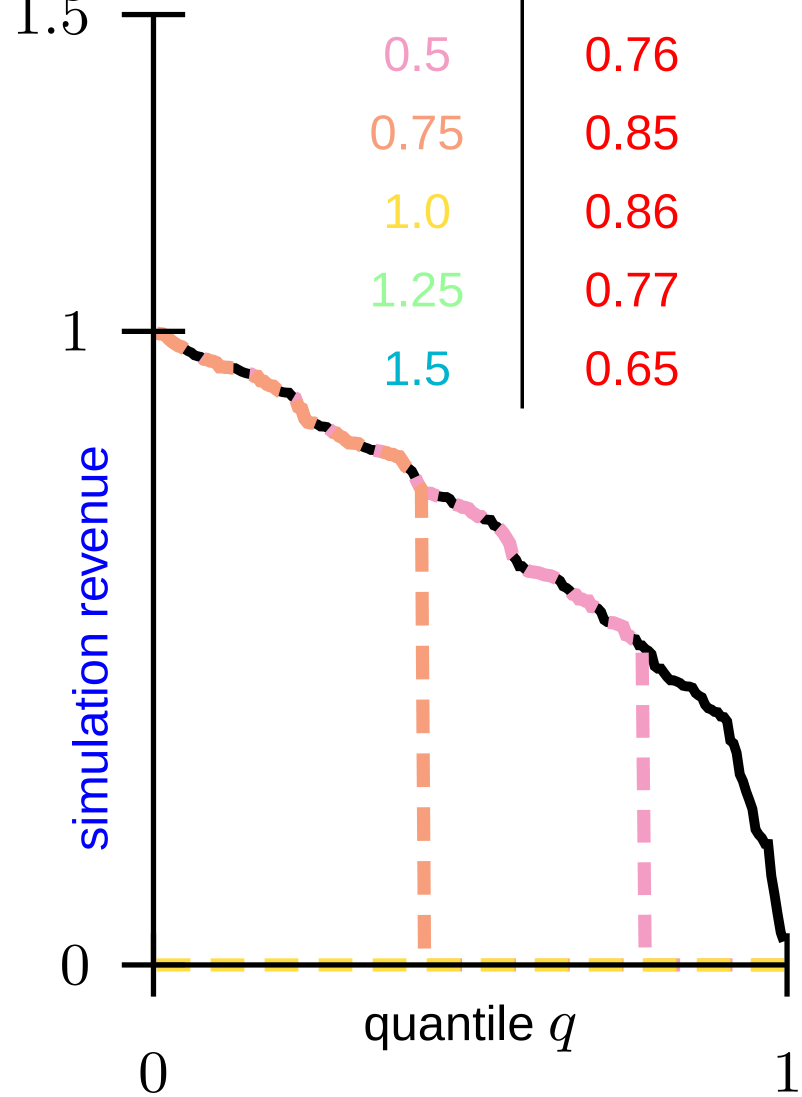

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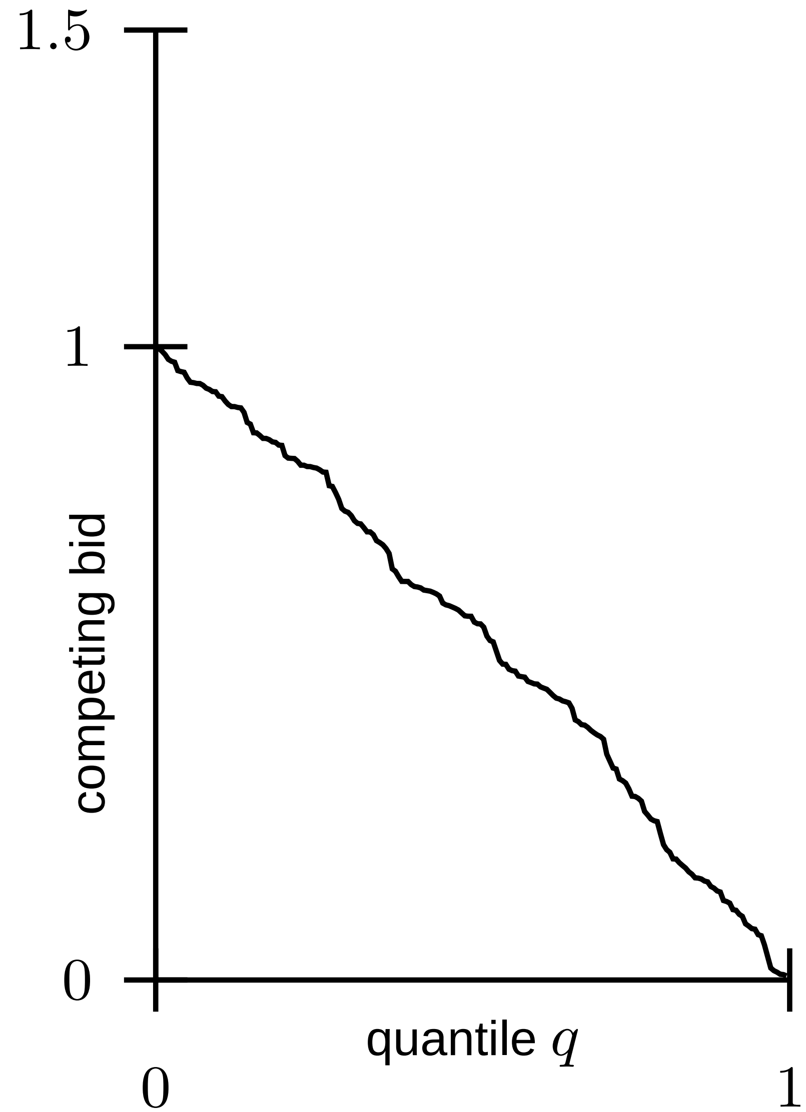

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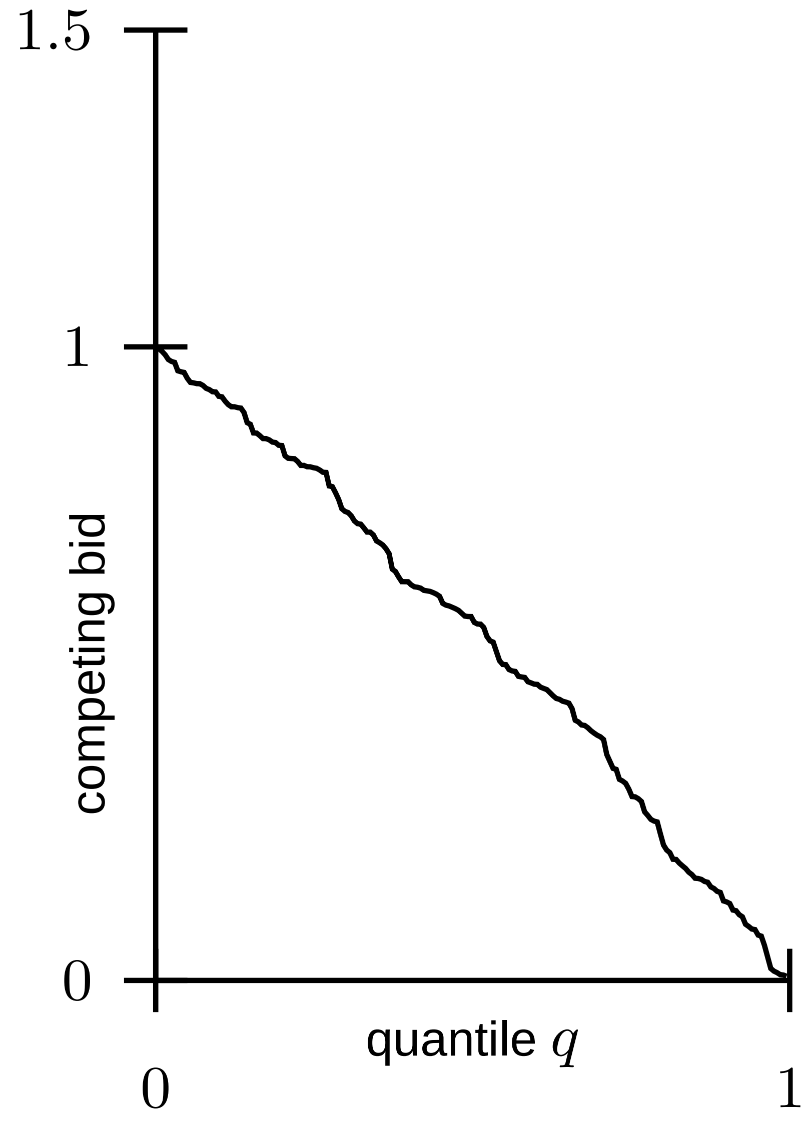

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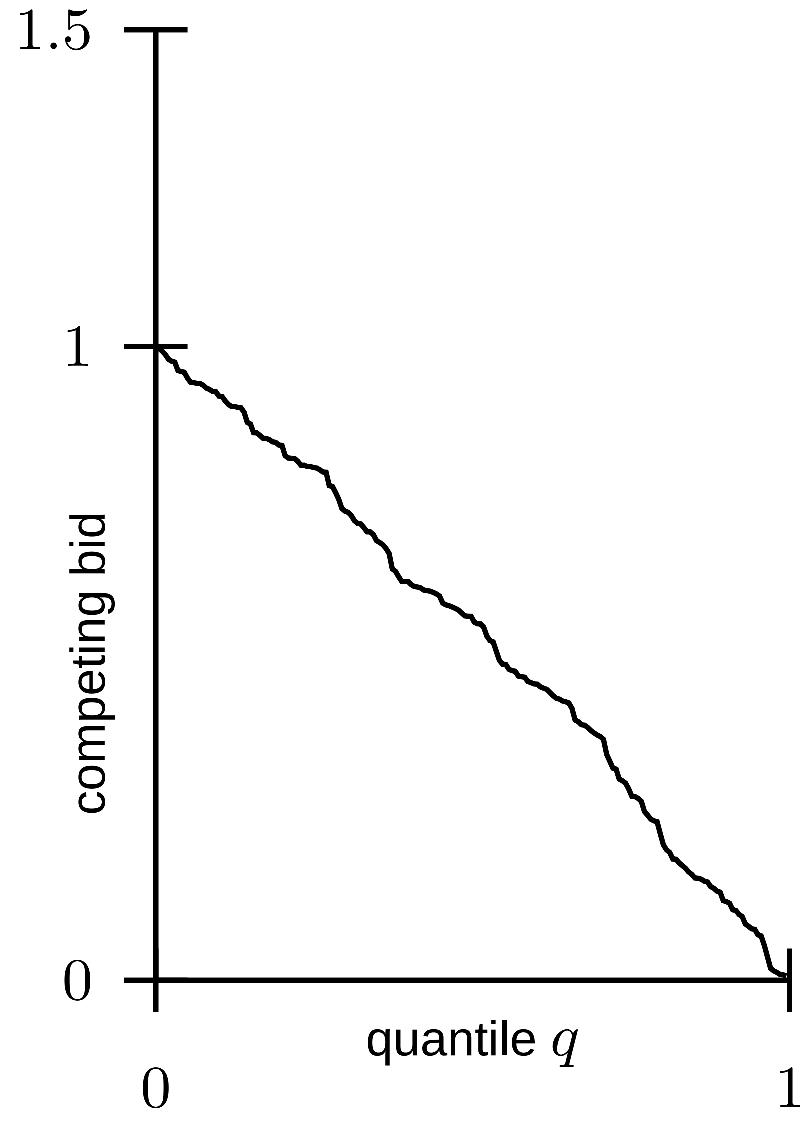
Approach:

1. given bid distribution, solve for bid strategy
2. invert bid strategy to get bidder's value for item from bid.



Bidder's Bid Optimization

Example: two bidders, first-price auction.

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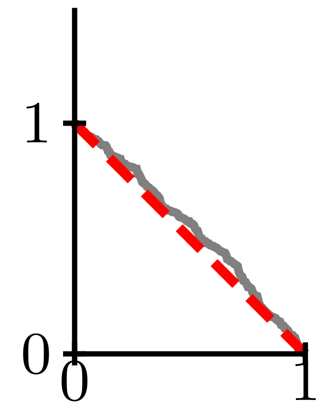

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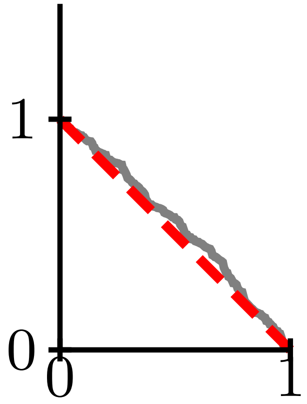

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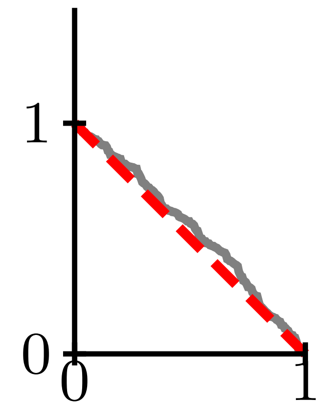

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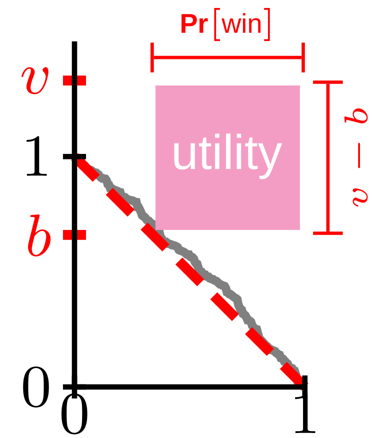

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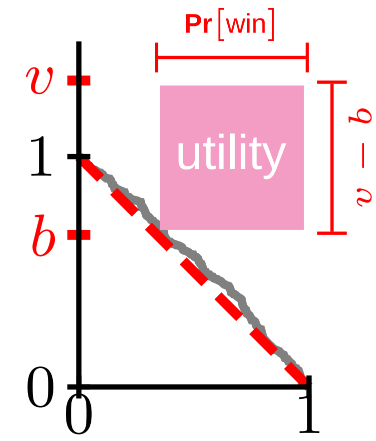

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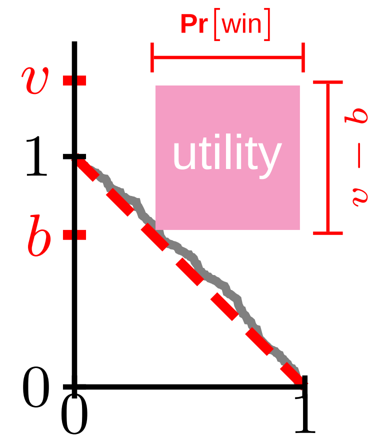

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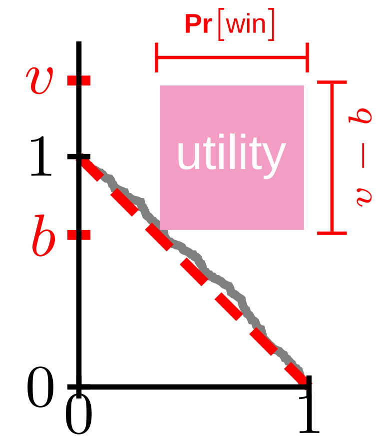

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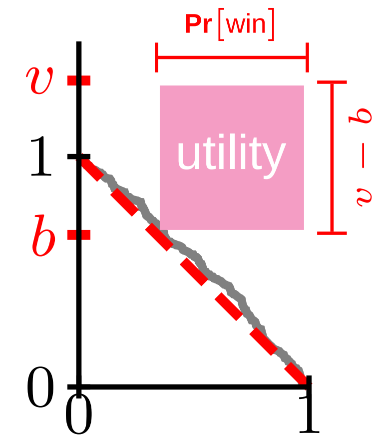

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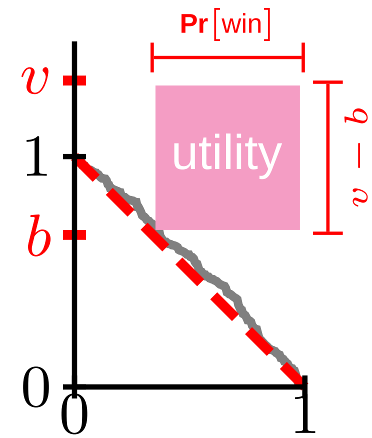
Conclusion 2: So values are uniform on $[0, 2]$.



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Conclusion 2: So values are uniform on $[0, 2]$.
Conclusion 3: From value distribution can solve for equilibrium behavior in any auction!

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Inference Equation: for first price auction

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Questions?

Research Directions

Research Directions:

- are there simple mechanisms that are approximately optimal?
  (e.g., price of anarchy or price of stability)
- is the optimal mechanism tractible to compute (even if it is complex)?
- what are optimal auctions for multi-dimensional agent preferences?
- what are the optimal auctions for non-linear agent preferences, e.g., from budgets or risk-aversion?
- are there good mechanisms that are less dependent on distributional assumptions?

BNE and Auction Theory Homework

1. For two agents with values $U[0,1]$ and $U[0,2]$, respectively:
   (a) show that the first-price auction is not socially optimal in BNE.
   (b) give an auction with "pay your bid if you win" semantics that is.
2. What is the virtual value function for an agent with value $U[0,2]$?
3. What is revenue optimal single-item auction for:
   (a) two agents with values $U[0,2]$? n agents?
   (b) two agents with values $U[a,b]$?
   (c) two values $U[0,1]$ and $U[0,2]$, respectively?
4. For n agents with values $U[0,1]$ and a *public good*, i.e., where either all or none of the agents can be served,
   (a) What is the revenue optimal auction?
   (b) What is the expected revenue of the optimal auction? (use big-oh notation)

`http://jasonhartline.com/MDnA/`