BNE and Auction Theory Homework

1. For two agents with values $U[0,1]$ and $U[0,2]$, respectively:
   (a) show that the first-price auction is not socially optimal in BNE.
   (b) give an auction with "pay your bid if you win" semantics that is.
2. What is the virtual value function for an agent with value $U[0,2]$?
3. What is revenue optimal single-item auction for:
   (a) two agents with values $U[0,2]$? n agents?
   (b) two agents with values $U[a,b]$?
   (c) two values $U[0,1]$ and $U[0,2]$, respectively?
4. For n agents with values $U[0,1]$ and a *public good*, i.e., where either all or none of the agents can be served,
   (a) What is the revenue optimal auction?
   (b) What is the expected revenue of the optimal auction? (use big-oh notation)

`http://jasonhartline.com/MDnA/`

Bayesian Mechanism Design

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July 28, 2014

Vignettes from Manuscript
Mechanism Design and Approximation
http://jasonhartline.com/MDnA/

Goals for Mechanism Design Theory

Mechanism Design: how can a social planner / optimizer achieve objective when participant preferences are private.

Goals for Mechanism Design Theory:

- *Descriptive:* predict/affirm mechanisms arising in practice.
- *Prescriptive:* suggest how good mechanisms can be designed.
- *Conclusive:* pinpoint salient characteristics of good mechanisms.
- *Tractable:* mechanism outcomes can be computed quickly.

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Informal Thesis: *approximately optimality* is often descriptive, prescriptive, conclusive, and tractable.

Example 1: Gambler's Stopping Game

A Gambler's *Stopping Game*:

- *sequence* of n games,
- *prize* of game i is distributed from F_i,
- *prior-knowledge* of distributions.

On day i, gambler plays game i:

- *realizes* prize $v_i \sim F_i$,
- chooses to keep prize and *stop*, or
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Question: How should our gambler play?

Optimal Strategy

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Discussion:

- *Complicated:* n different, unrelated thresholds.
- *Inconclusive:* what are properties of good strategies?
- *Non-robust:* what if order changes? what if distribution changes?
- *Non-general:* what do we learn about variants of Stopping Game?

Threshold Strategies and Prophet Inequality

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$$\mathbf{E}[\text{prize for strategy } t] \geq \mathbf{E}[\max_i v_i]/2.$$

[Samuel-Cahn '84]

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Discussion:

- *Simple:* one number t.
- *Conclusive:* trade-off "stopping early" with "never stopping".
- *Robust:* change order? change distribution above or below t?
- *General:* same solution works for similar games: invariant of "tie-breaking rule"

Prophet Inequality Proof

0. Notation:

- $q_i = \textbf{Pr}[v_i < t]$.
- $x = \textbf{Pr}[\text{never stops}] = \prod_i q_i$.

1. Upper Bound on $\textbf{E}[\max]$:

2. Lower Bound on $\textbf{E}[\text{prize}]$:

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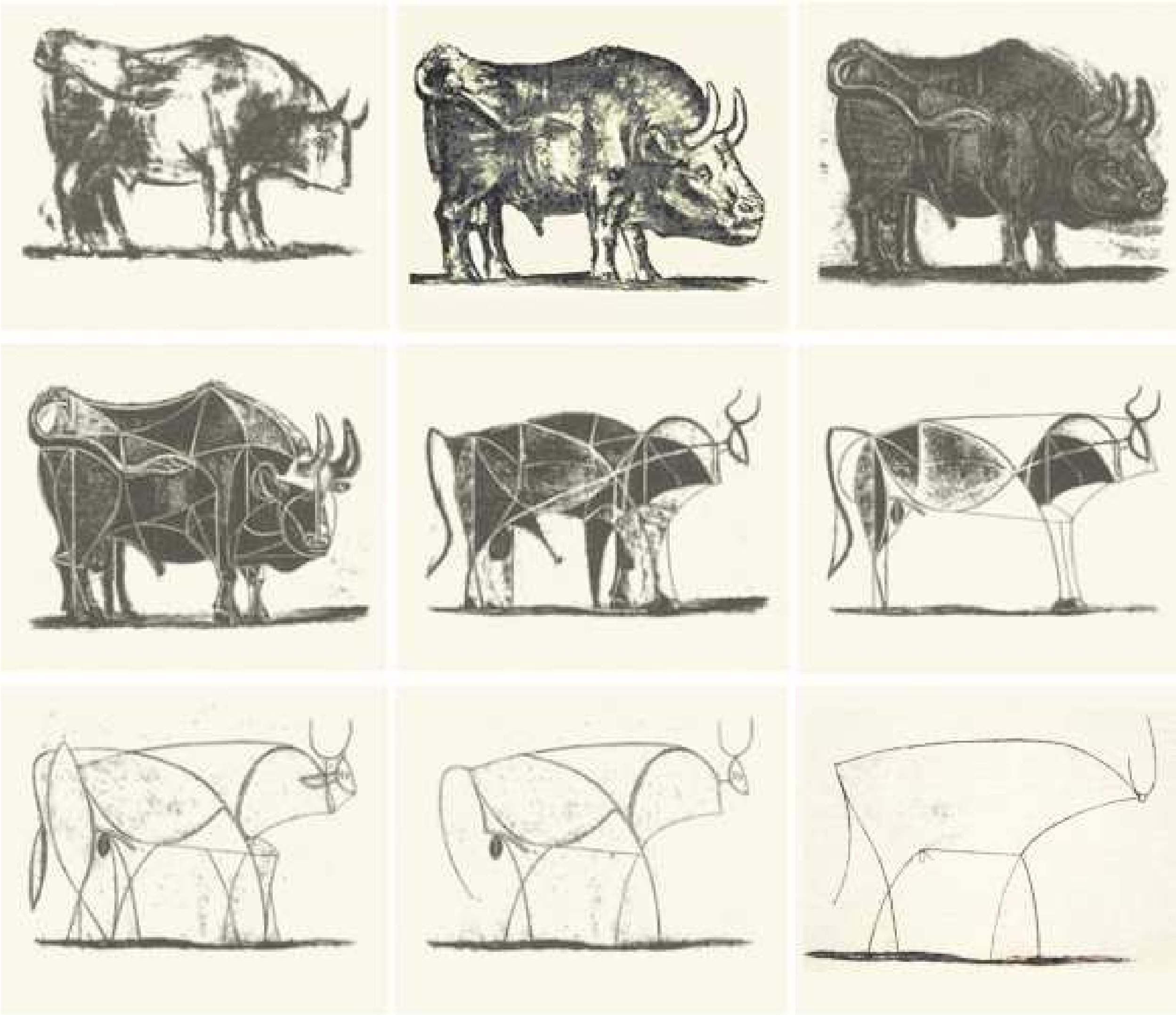
- gives relevant intuition for practice
- gives simple, robust solutions.
- Exact optimization is often impossible.
  (information theoretically, computationally, analytically)

Picasso



[Picasso's Bull 1945–1946 (one month)]

Questions?

Overview

Part I: Optimal Mechanism Design

- single-item auction.
- objectives: social welfare vs. seller profit.
- characterization of Bayes-Nash equilibrium.
- consequences: solving, uniqueness, and optimizing over BNE.
- inferring values from bids.

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- single-item auctions.
- multi-dimensional auctions.
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- single-item auctions. (Chapter 4)
- multi-dimensional auctions. (Chapter 7)
- prior-independent auctions. (Chapter 5)
- computationally tractable mechanisms. (Chapter 8)

Part IIa: Approximation for single-dimensional Bayesian mechanism design

(where agent preferences are given by a private value for service, zero for no service; preferences are drawn from a distribution)

Example 2: Single-item auction

Problem: Bayesian Single-item Auction Problem

- a single item for sale,
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- a dist. $\mathbf{F} = F_1 \times \cdots \times F_n$ from which the consumers' values for the item are drawn.

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Question: What is optimal auction?

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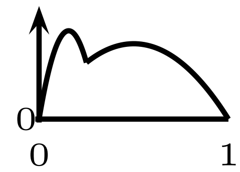

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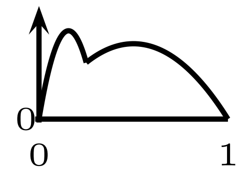

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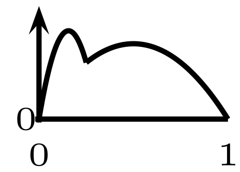

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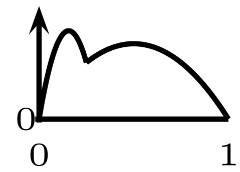

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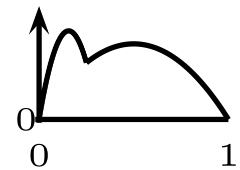

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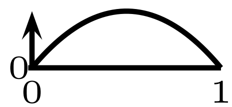

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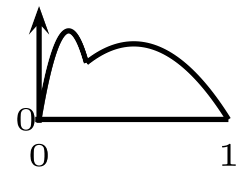

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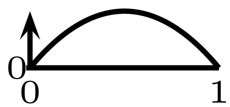

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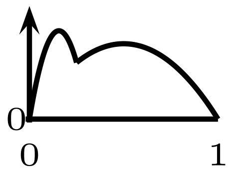

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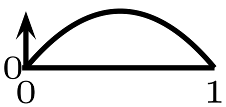

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8. **Cor:** for iid, regular dists, optimal auction is *second-price with reserve price* $\varphi^{-1}(0)$.

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Discussion:

- iid, regular case: seems very special.
- general case: optimal auction rarely used. (too complicated?)

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threshold t	virtual price
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Discussion:

- constant virtual price $\Rightarrow$ bidder-specific reserves.
- *simple:* reserve prices natural, practical, and easy to find.
- *robust:* posted pricing with arbitrary tie-breaking works fine, collusion fine, etc.

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Question: for non-identical distributions, is *anonymous reserve* approximately optimal?

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Discussion:

- theorem is not tight, actual bound is in $[2, 4]$.
- justifies wide prevalence.

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Thm: non-identical (possibly irregular) distributions, *posted pricing mechanisms* are often constant approximations.

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Proof technique:

- optimal mechanism is a virtual surplus maximizer.
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Basic Open Question: to what extent do simple mechanisms approximate (well understood but complex) optimal ones?

Challenges: non-downward-closed settings, negative virtual values.

Questions?

Part IIb: Approximation for multi-dimensional Bayesian mechanism design

(where agent preferences are given by values for each available service, zero for no service; preferences drawn from distribution)

Example 3: unit-demand pricing

Problem: Bayesian Unit-Demand Pricing

- a single, unit-demand consumer.
- n items for sale.
- a dist. $\mathbf{F} = F_1 \times \cdots \times F_n$ from which the consumer's values for each item are drawn.

Goal: seller optimal *item-pricing* for $\mathbf{F}$.

Example 3: unit-demand pricing

Problem: Bayesian Unit-Demand Pricing

- a single, unit-demand consumer.
- n items for sale.
- a dist. $\mathbf{F} = F_1 \times \cdots \times F_n$ from which the consumer's values for each item are drawn.

Goal: seller optimal *item-pricing* for $\mathbf{F}$.

Question: What is optimal pricing?

Optimal Pricing

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Discussion:

- little conceptual insight and
- not generally tractable.

Analogy

Challenge: approximate optimal but we do not understand it?

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Problem: Bayesian Unit-demand Pricing (a.k.a., MD-PRICING)

- a single, *unit-demand* buyer,
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Problem: Bayesian Single-item Auction (a.k.a., SD-AUCTION)

- a single item for sale,
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Proof: prophet inequality (tie-break by "$-p_i$"). [Chawla,Hartline,Malec,Sivan'10]

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   (virtual surplus approximation)

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Discussion:

- *robust* to agent ordering, collusion, etc.
- *conclusive*:
  - competition not important for approximation.
  - unit-demand incentives similar to single-dimensional incentives.
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Open Question: identify upper bounds beyond unit-demand settings:

- analytically tractable and
- approximable.

Questions?

Part IIc: Approximation for prior-independent mechanism design.

(mechanisms should be good for any set of agent preferences, not just given distributional assumptions)

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- what if one mechanism must be used in many scenarios?

Question: can we design good auctions without knowledge of prior-distribution?

Optimal Prior-independent Mechs

Optimal Prior-indep. Mech: (a.k.a., non-parametric implementation)

1. agents report value and prior,
2. shoot agents if disagree, otherwise
3. run optimal mechanism for reported prior.

Discussion:

- *complex*, agents must report high-dimensional object.
- *non-robust*, e.g., if agents make mistakes.
- *inconclusive*, begs the question.

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- “recruit one more bidder” is prior-independent strategy.
- “bicriteria” approximation result.
- *conclusive:* competition more important than optimization.

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- "recruit one more bidder" is prior-independent strategy.
- "bicriteria" approximation result.
- *conclusive:* competition more important than optimization.
- *non-general*: e.g., for k-unit auctions, need k additional bidders.

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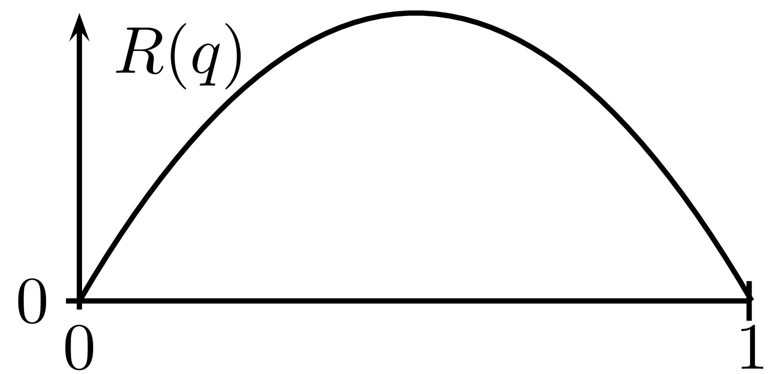
$$R(q) = q \cdot F^{-1}(1-q)$$



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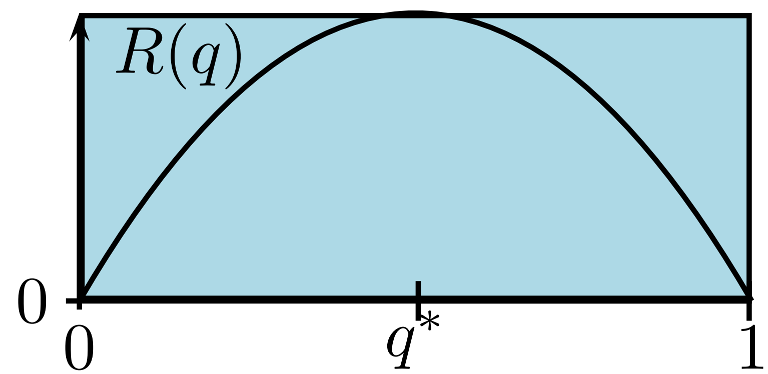

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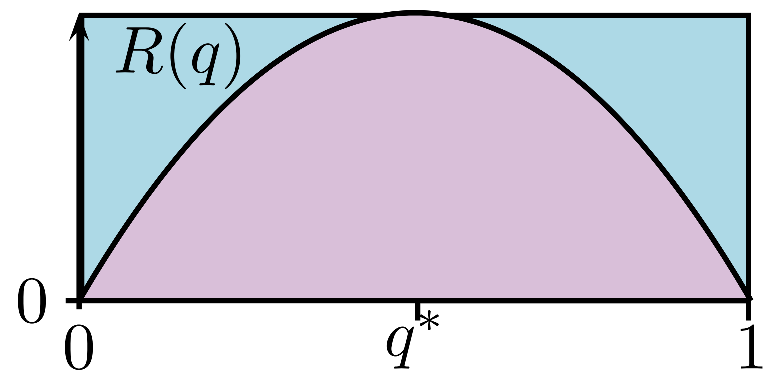

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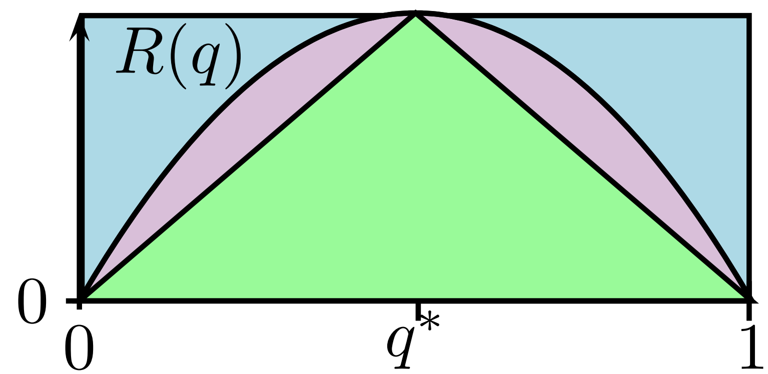

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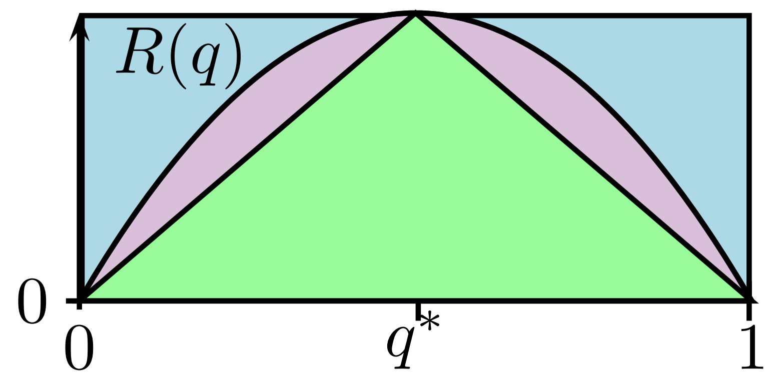

- So second-price on two bidders $\geq$ optimal revenue on one bidder.

Example 4: digital goods

Question: how should a profit-maximizing seller sell a *digital good* (n bidder, n copies of item)?

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Discussion:

- optimal,
- simple, but
- not prior-independent

Approximation via Single Sample

Single-Sample Auction: (for digital goods)

[Dhangwatnotai, Roughgarden, Yan '10]

1. pick random agent i as sample.
2. offer all other agents price v_i.
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Proof: from geometric argument.

Discussion:

- *prior-independent*.
- *conclusive*,
  - learn distribution from reports, not cross-reporting.
  - don't need precise distribution, only need single sample for approximation. (more samples can improve approximation/robustness.)
- *generic*, applies to general settings.

Recent Extensions:

- non-identical distributions. [Dhangwatnotai, Roughgarden, Yan ’10]
- position auctions, matroids, downward-closed environments. [Hartline, Yan ’11; Ha, Hartline ’11]
- multi-item auctions (multi-dimensional preferences). [Devanur, Hartline, Karlin, Nguyen ’11; Roughgarden, Talgam-Cohen, Yan ’12]

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Questions?

Part IId: Computational Tractability in Bayesian mechanism design

(where the optimal mechanism may be computationally intractable)

Example 5: single-minded combinatorial auction

Problem: Single-minded combinatorial auction

- n agents,
- m items for sale.
- Agent i wants only bundle $S_i \subset \{1, \ldots, m\}$.
- Agent i's value v_i drawn from F_i.

Goal: auction to maximize *social surplus* (a.k.a., welfare).

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Optimal Combinatorial Auction: Vickrey-Clarke-Groves (VCG):

1. allocate to maximize reported surplus,
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Discussion:

- distribution is irrelevant (for welfare maximization).
- Step 1 is NP-hard *weighted set packing* problem.
- Cannot replace Step 1 with approximation algorithm.

BNE reduction

Question: Can we convert any algorithm into a mechanism without reducing its social welfare?

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- Run $\mathcal{A}(\sigma_1(v_1), \ldots, \sigma_n(v_n))$.
- σ_i calculated from *max weight matching* on i's type space.
  - stationary with respect to F_i.
  - $x_i(\sigma_i(v_i))$ monotone.
  - welfare preserved.

Example: σ_i

Example:

$F_i(v_i)$	v_i	$x_i(v_i)$
.25	1	0.1
.25	4	0.5
.25	5	0.4
.25	10	1.0

Example:

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Note:

- σ_i is from max weight matching between v_i and $x_i(v_i)$.
- σ_i is stationary.
- σ_i (weakly) improves welfare.

BNE reduction discussion

Thm: Any algorithm can be converted into a mechanism with no loss in expected welfare. Runtime is polynomial in size of agent's type space.
[Hartline, Lucier '10; Hartline, Kleinberg, Malekian '11; Bei, Huang '11]

Discussion:

- applies to all algorithms not just worst-case approximations.
- BNE incentive constraints are solved independently.
- works with multi-dimensional preferences too.

Extensions

Extension:

- impossibility for dominant strategy reduction.

[Chawla, Immorlica, Lucier ’12]

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Open Questions:

- non-brute-force in type-space? e.g., for product distributions?
- other objectives, e.g., makespan? [Chawla, Immorlica, Lucier ’12]

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- approximation pinpoints salient characteristics of good mechanisms.
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- posted-pricings are approximately optimal.
- good mechanisms can be designed without prior information.
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